

Optimal ground state energy of two-phase conductors

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Abstract. Consider the problem of distributing two conducting materials in a ball with fixed proportion in order to minimize the first eigenvalue of a Dirichlet operator. It was conjectured that the optimal distribution consists of putting the material with the highest conductivity in a ball around the center. In this paper, we show that the conjecture is false for all dimensions $n \geq 2$.

Key Words: Eigenvalue optimization; Two-phase conductors; Rearrangements; Bessel functions¹

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^n with a smooth boundary which is to be called the design region and consider two conducting materials with conductivities $0 < \alpha < \beta$. These materials are distributed in Ω such that the volume of the region D occupied by the material with conductivity β is a fixed number A with $0 < A < |\Omega|$. Consider the following two-phase eigenvalue problem

$$-\operatorname{div}((\beta\chi_D + \alpha\chi_{D^c})\nabla u) = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where $\beta\chi_D + \alpha\chi_{D^c}$ is the conductivity, λ is the ground state energy or the smallest positive eigenvalue and u is the corresponding eigenfunction.

We use notation $\lambda(D)$ to show the dependence of the eigenvalue on D , the region with the highest conductivity. To determine the system's profile which gives the minimum principal eigenvalue, we should verify the following optimization problem

$$\inf_{D \subset \Omega, |D|=A} \lambda(D), \quad (1.2)$$

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where λ has the following variational formulation

$$\lambda(D) = \min_{u \in H_0^1(\Omega), \|u\|_{L^2(\Omega)}=1} \int_{\Omega} (\beta \chi_D + \alpha \chi_{D^c}) |\nabla u|^2 dx. \quad (1.3)$$

In general, this problem has no solution in any class of usual domains. Cox and Lipton have given in [9] conditions for an optimal microstructural design. However, when Ω is a ball, the symmetry of the domain implies that there exists a radially symmetric minimizer. Alvino *et al* have obtained this result thanks to a comparison result for Hamilton-Jacobi equations [1]. Conca *et al.* have revived interest in this problem by giving a new simpler proof of the existence result only using rearrangement techniques [8].

In eigenvalue optimization for elliptic partial differential equations, one of challenging mathematical problems after the problem of existence is an exact formula of the optimizer or optimal shape design. Most papers in this field answered this question just in case Ω is a ball [5, 10, 12, 18, 17]. This class of problems is difficult to solve due to the lack of the topology information of the optimal shape. For one-dimensional case, Krein has shown in [14] that the unique minimizer of (1.2) is obtained by putting the material with the highest conductivity in an interval in the middle of the domain. Surprisingly, the exact distribution of the two materials which solves optimization problem (1.2) is still not known for higher dimensions.

Let $\Omega = \mathcal{B}(0, \mathcal{R})$ be a ball centered at the origin with radius \mathcal{R} , the solution of the one-dimensional problem suggests for higher dimensions that $\mathcal{B}(0, \mathcal{R}^*)$ is a natural candidate to be the optimal domain. This conjecture has been supported by numerical evidence in [7] using the shape derivative analysis of the first eigenvalue for the two-phase conduction problem. In addition, it has been shown in [11] employing the second order shape derivative calculus that $D = \mathcal{B}(0, \mathcal{R}^*)$ is a local strict minimum for the optimization problem (1.2) when A is small enough. In spite of the above evidence, it has been established in [6] that the conjecture is not true in two- or three- dimensional spaces when α and β are close to each other (low contrast regime) and A is sufficiently large. The theoretical base for the result is an asymptotic expansion of the eigenvalue with respect to $\beta - \alpha$ as $\beta \rightarrow \alpha$, which allows one to approximate the optimization problem by a simple minimization problem.

In this paper, we investigate the conjecture for all dimensions $n \geq 2$. We prove that the conjecture is false not only for two- or three- dimensional spaces, but also for all dimensions $n \geq 2$. We have provided a different proof of the main result in [6] and we will establish it in a vastly simpler way.

2 Preliminaries

In order to establish the main theorem, we need some preparation. Our proof is based upon the properties of Bessel functions. In this section, we state some results from the theory of Bessel functions. The reader can refer to [3, 21] for further information about Bessel functions.

Consider the standard form of Bessel equation which is given by

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \quad (2.1)$$

where ν is a nonnegative real number. The regular solution of (2.1), called the Bessel function of the first kind of order ν , is given by

$$J_{\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k+\nu} \Gamma(\nu + k + 1)}, \quad (2.2)$$

where Γ is the gamma function. We shall use following recurrence relations between Bessel functions

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x), \quad (2.3)$$

$$(x^{-\nu} J_{\nu}(x))' = -x^{-\nu} J_{\nu+1}(x). \quad (2.4)$$

Let $j_{\nu,m}$ be the m th positive zeros of the function $J_{\nu}(x)$, then it is well known that the zeros of $J_{\nu}(x)$ are simple with possible exception of $x = 0$. In addition, we have the following lemma related to the roots of $J_{\nu}(x)$, [3, 21].

Lemma 2.1. *When $\nu \geq 0$, the positive roots of $J_{\nu}(x)$ and $J_{\nu+1}(x)$ interlace according to the inequalities*

$$j_{\nu,m} < j_{\nu+1,m} < j_{\nu,m+1}.$$

We will need the following technical assertion later.

Lemma 2.2. *If $\nu_1, \nu_2 \geq 0$, then*

$$(\nu_2^2 - \nu_1^2) \int_0^{\tau} \frac{J_{\nu_2}(s) J_{\nu_1}(s)}{s} ds = \tau (J'_{\nu_2}(\tau) J_{\nu_1}(\tau) - J_{\nu_2}(\tau) J'_{\nu_1}(\tau)).$$

Proof. Functions J_{ν_2} and J_{ν_1} are solutions of Bessel equations

$$x^2 J''_{\nu_2} + x J'_{\nu_2} + (x^2 - \nu_2^2) J_{\nu_2} = 0,$$

$$x^2 J''_{\nu_1} + x J'_{\nu_1} + (x^2 - \nu_1^2) J_{\nu_1} = 0.$$

Multiplying the first equation by J_{ν_1} and the second one by J_{ν_2} , we have

$$\frac{\nu_2^2}{x} J_{\nu_2} J_{\nu_1} = x J''_{\nu_2} J_{\nu_1} + J'_{\nu_2} J_{\nu_1} + x J_{\nu_2} J'_{\nu_1},$$

$$\frac{\nu_1^2}{x} J_{\nu_2} J_{\nu_1} = x J''_{\nu_1} J_{\nu_2} + J'_{\nu_1} J_{\nu_2} + x J_{\nu_2} J'_{\nu_1}.$$

Subtracting the second equality from the first one,

$$[x(J'_{\nu_2} J_{\nu_1} - J'_{\nu_1} J_{\nu_2})]' = \frac{(\nu_2^2 - \nu_1^2)}{x} J_{\nu_2} J_{\nu_1}.$$

Integrating this equation from 0 to τ , leads to the assertion. \square

This section is closed with some results from the rearrangement theory related to our optimization problems. The reader can refer to [1, 4] for further information about the theory of rearrangements.

Definition 2.1. Two Lebesgue measurable functions $\rho : \Omega \rightarrow \mathbb{R}$, $\rho_0 : \Omega \rightarrow \mathbb{R}$, are said to be rearrangements of each other if

$$|\{x \in \Omega : \rho(x) \geq \tau\}| = |\{x \in \Omega : \rho_0(x) \geq \tau\}| \quad \forall \tau \in \mathbb{R}. \quad (2.5)$$

The notation $\rho \sim \rho_0$ means that ρ and ρ_0 are rearrangements of each other. Consider $\rho_0 : \Omega \rightarrow \mathbb{R}$, the class of rearrangements generated by ρ_0 , denoted \mathcal{P} , is defined as follows

$$\mathcal{P} = \{\rho : \rho \sim \rho_0\}.$$

Let $\rho_0 = \beta \chi_{D_0} + \alpha \chi_{D_0^c}$ where $D_0 \subset \Omega$ and $|D_0| = A$. For the sake of completeness, we include following technical assertion.

Lemma 2.3. *A function ρ belongs to the rearrangement class \mathcal{P} if and only if $\rho = \beta\chi_D + \alpha\chi_{D^c}$ such that $D \subset \Omega$ and $|D| = A$.*

Proof. Assume $\rho \in \mathcal{P}$. In view of definition 2.1,

$$\begin{aligned} |\{x \in \Omega : \rho_0(x) = r\}| &= |\cap_1^\infty \{x \in \Omega : r \leq \rho_0(x) < r + \frac{1}{n}\}| \\ &= \lim_{n \rightarrow \infty} |\{x \in \Omega : \rho_0(x) \geq r\}| - |\{x \in \Omega : \rho_0(x) \geq r + \frac{1}{n}\}| \\ &= \lim_{n \rightarrow \infty} |\{x \in \Omega : \rho(x) \geq r\}| - |\{x \in \Omega : \rho(x) \geq r + \frac{1}{n}\}| \\ &= |\cap_1^\infty \{x \in \Omega : r \leq \rho(x) < r + \frac{1}{n}\}| = |\{x \in \Omega : \rho(x) = r\}|, \end{aligned}$$

where it means that the level sets of ρ and ρ_0 have the same measures and this yields the assertion. The other part of the theorem is concluded from definition 2.1. \square

Let us state here one of the essential tools in studying rearrangement optimization problems.

Lemma 2.4. *Let \mathcal{P} be the set of rearrangements of a fixed function $\rho_0 \in L^r(\Omega)$, $r > 1$, $\rho_0 \not\equiv 0$, and let $q \in L^s(\Omega)$, $s = r/(r-1)$, $q \not\equiv 0$. If there is a decreasing function $\eta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\eta(q) \in \mathcal{P}$, then*

$$\int_{\Omega} \rho q dx \geq \int_{\Omega} \eta(q) q dx \quad \forall \rho \in \mathcal{P},$$

and the function $\eta(q)$ is the unique minimizer relative to \mathcal{P} .

Proof. See [4]. \square

3 Refusing the conjecture

In this section, we investigate the conjecture proposed in [7] when Ω is a ball in \mathbb{R}^n such that $n \geq 2$. We show that the conjecture is false not only for $n = 2, 3$ but also for every $n \geq 4$. Indeed, we will establish that a ball could not be a global minimizer for the optimization problem (1.2) when α and β are close to each other (low contrast regime) and A is large enough. It should be noted that our method is not as complicated as the approach has been stated in [6] and we deny the conjecture in a simpler way.

We hereafter regard $\Omega \subset \mathbb{R}^n$ as the unit ball centered at the origin. Assume that ψ is the eigenfunction corresponding to the principal eigenvalue of the Laplacian with Dirichlet's boundary condition on Ω . Then, one can consider $\psi = \psi(r)$ as a radial function which satisfies

$$\begin{cases} r^2 \psi''(r) + (n-1)r\psi'(r) + \lambda r^2 \psi(r) = 0 & 0 < r < 1, \\ \psi'(0) = 0 & \psi(1) = 0, \end{cases} \quad (3.1)$$

where the boundary conditions correspond to the continuity of the gradient at the origin and Dirichlet's condition on the boundary. In the next lemma, we examine the function $|\psi'(r)|$.

Lemma 3.1. *Let ψ be the eigenfunction of (3.1) associated with the principal eigenvalue λ . Then, function $|\psi'(r)|$ has a unique maximum point ρ_n in $(0, 1)$.*

Proof. The solution of (3.1) is

$$\psi(r) = r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\mu r) \quad 0 \leq r \leq 1,$$

where $\mu = j_{\frac{n}{2}-1,1}$. For the reader's convenience, we use the change of variable $t = \mu r$ and then

$$\psi(t) = \mu^{\frac{n}{2}-1} \left(\frac{J_{\frac{n}{2}-1}(t)}{t^{\frac{n}{2}-1}} \right) \quad 0 \leq t \leq \mu.$$

According to lemma 2.1, $j_{\frac{n}{2}-1,1} < j_{\frac{n}{2},1}$ and then we see $J_{\frac{n}{2}}(t) \geq 0$ for $0 \leq t \leq \mu$. Therefore,

$$|\psi'(t)| = \mu^{\frac{n}{2}-1} \left(\frac{J_{\frac{n}{2}}(t)}{t^{\frac{n}{2}-1}} \right) \quad 0 \leq t \leq \mu,$$

invoking formula (2.4). To determine the maximum point of this function, one should calculate $\frac{d}{dt}(|\psi'(t)|)$. Employing relations (2.3) and (2.4),

$$\frac{d}{dt}(|\psi'(t)|) = \frac{\mu^{\frac{n}{2}-1} (t J_{\frac{n}{2}-1}(t) - (n-1) J_{\frac{n}{2}}(t))}{t^{\frac{n}{2}}}.$$

Then $\frac{d}{dt}(|\psi'(t)|) = 0$ yields

$$t J_{\frac{n}{2}-1}(t) - (n-1) J_{\frac{n}{2}}(t) = 0.$$

The zeros of the last equation are the fixed points of the function

$$g(t) = (n-1) \frac{J_{\frac{n}{2}}(t)}{J_{\frac{n}{2}-1}(t)} \quad 0 < t < \mu.$$

We find that

$$J_{\frac{n}{2}}'(t) J_{\frac{n}{2}-1}(t) - J_{\frac{n}{2}}(t) J_{\frac{n}{2}-1}'(t) = \frac{(n-1)}{t} \int_0^t \frac{J_{\frac{n}{2}}(\tau) J_{\frac{n}{2}-1}(\tau)}{\tau} d\tau,$$

applying lemma 2.2. Consequently, $g'(t) > 0$ for $0 < t < \mu$ and g is an increasing function. On the other hand, $g(t)$ tends to infinity when $t \rightarrow \mu$ and, in view of formula (2.2), it tends to zero when $t \rightarrow 0$. Thus, $g(t)$ has a unique fixed point ρ_n in $(0, \mu)$ which it is the unique extremum point of $|\psi'(t)|$. Recall that $t J_{\frac{n}{2}-1}(t) - (n-1) J_{\frac{n}{2}}(t)$ is negative when $t \rightarrow \mu$. Hence, $\frac{d}{dt}(|\psi'(t)|)$ is negative in a neighborhood of μ and thus, ρ_n is the unique maximum point of $\frac{d}{dt}(|\psi'(t)|)$ in $(0, \mu)$. \square

We need the following theorem to deduce the main result.

Theorem 3.2. Assume D_0 is a subset of Ω where $|D_0| = A$ and u_0 is the eigenfunction of (1.1) corresponding to $\lambda(D_0)$. Let D_1 be a subset of Ω where

$$|D_1| = A \text{ and } D_1 = \{x : |\nabla u_0| \leq t\} \quad (3.2)$$

with

$$t = \inf\{s \in \mathbb{R} : |\{x : |\nabla u_0| \leq s\}| \geq A\}. \quad (3.3)$$

Then, $\lambda(D_1) \leq \lambda(D_0)$.

Proof. It is well known, from the Krein-Rutman theorem [15], that u_0 is positive everywhere on Ω . Therefore, we infer that all sets $\{x : |\nabla u_0| = s\}$ have measure zero because of lemma 7.7 in [13]. Then, one can determine set D_1 uniquely using the above formula. Let us define the following decreasing function

$$\eta(s) = \begin{cases} \beta & 0 \leq s \leq t^2, \\ \alpha & s > t^2, \end{cases}$$

where it yields

$$\eta(|\nabla u_0|^2) = \beta\chi_{D_1} + \alpha\chi_{D_1^c}.$$

Employing lemma 2.3 and 2.4, we can deduce

$$\int (\beta\chi_{D_1} + \alpha\chi_{D_1^c})|\nabla u_0|^2 dx \leq \int (\beta\chi_{D_0} + \alpha\chi_{D_0^c})|\nabla u_0|^2 dx,$$

and then we have $\lambda(D_1) \leq \lambda(D_0)$ invoking (1.3). \square

Remark 3.1. In theorem 3.2, if $D_1 \neq D_0$, then

$$\int (\beta\chi_{D_1} + \alpha\chi_{D_1^c})|\nabla u_0|^2 dx < \int (\beta\chi_{D_0} + \alpha\chi_{D_0^c})|\nabla u_0|^2 dx,$$

applying the uniqueness of the minimizer in lemma 2.4. Thus, we observe that $\lambda(D_1) < \lambda(D_0)$ when $D_1 \neq D_0$.

Remark 3.2. In [6], it has been proved that if $\rho_* = \beta\chi_{D_*} + \alpha\chi_{D_*^c}$ is the minimizer of

$$\min_{\rho \in \mathcal{P}} \int_{\Omega} \rho |\nabla \psi|^2 dx, \quad (3.4)$$

then the set D_* is an approximate solution for (1.2), under the assumption of low contrast regime. By arguments similar to those in the proof of theorem 3.2, one can determine the unique minimizer of problem (3.4), $\rho_* = \beta\chi_{D_*} + \alpha\chi_{D_*^c}$, using formulas (3.2) and (3.3). Recall from lemma 3.1 that $|\psi'(r)|$ has a unique maximum point ρ_n in $(0, 1)$ and it is a continuous function on $[0, 1]$ with $|\psi'(0)| = 0$. Then the unique symmetrical domain D_* which $\rho_* = \beta\chi_{D_*} + \alpha\chi_{D_*^c}$ is the solution of (3.4) is of two possible types. The set D_* is a ball centered at the origin if $A \leq |\mathcal{B}(0, \rho_n)|$ and it is the union of a ball and an annulus touching the outer boundary of Ω if $A > |\mathcal{B}(0, \rho_n)|$.

This result has been established in [6] for $n = 2, 3$.

Now we are ready to state the main result. Indeed, we establish that locating the material with the highest conductivity in a ball centered at the origin is not the minimal distribution since we can find another radially symmetric distribution of the materials which has a smaller basic frequency.

Theorem 3.3. *Let $D_0 = \mathcal{B}(0, \rho) \subset \Omega$ be a ball centered at the origin with $|D_0| = A$. If β is sufficiently close to α and $\rho > \rho_n$, then there is a set $D_1 \subset \Omega$ with $|D_1| = A$ containing a radially symmetric subset of D_0^c where $\lambda(D_1) < \lambda(D_0)$.*

Proof. Suppose u_0 is the eigenfunction of (1.1) associated with $\lambda = \lambda(D_0)$ such that $\|u_0\|_{L^2(\Omega)} = 1$. Utilizing theorem 3.2 and remark 3.1, we conclude $\lambda(D_1) < \lambda(D_0)$ provided

$$D_1 = \{x : |\nabla u_0| \leq t\}, \quad t = \inf\{s \in \mathbb{R} : |\{x : |\nabla u_0| \leq s\}| \geq A\},$$

and $D_0 \neq D_1$. One can observe that u_0 satisfies the following transmission problem

$$\begin{cases} -\beta \Delta v_1 = \lambda v_1 & \text{in } D_0 \\ -\alpha \Delta v_2 = \lambda v_2 & \text{in } D_0^c \\ v_1(x) = v_2(x) & \text{on } \partial D_0 \\ \beta \frac{\partial}{\partial \mathbf{n}} v_1 = \alpha \frac{\partial}{\partial \mathbf{n}} v_2 & \text{on } \partial D_0 \\ v_2(x) = 0 & \text{on } \partial \Omega, \end{cases} \quad (3.5)$$

where \mathbf{n} is the unit outward normal. According to the above representation, u_0 is an analytic function in the closure of sets D_0 and D_0^c employing the analyticity theorem [2].

We should assert that $D_0 \neq D_1$. To this end, let us note that u_0 is a radial function and so $u_0(x) = y(r)$, $r = \|x\|$, where the function y solves

$$\begin{cases} y''(r) + \frac{n-1}{r}y'(r) + \frac{\lambda}{\beta}y(r) = 0 & \text{in } (0, \rho) \\ y''(r) + \frac{n-1}{r}y'(r) + \frac{\lambda}{\alpha}y(r) = 0 & \text{in } (\rho, 1) \\ y(\rho^-) = y(\rho^+) \\ \beta y'(\rho^-) = \alpha y'(\rho^+) \\ y'(0) = 0, \quad y(1) = 0. \end{cases} \quad (3.6)$$

We introduce $y_1(r)$ and $y_2(r)$ as the solution of (3.6) in $[0, \rho]$ and $[\rho, 1]$ respectively. We claim that if

$$|y_2'(1)| < z = \max_{r \in [0, \rho]} |y_1'(r)|, \quad (3.7)$$

then D_1 contains a radially symmetric subset of D_0^c and so D_1 is not equal to D_0 .

Recall that level sets of $|\nabla u_0|$ have measure zero. Hence, if $|y_2'(r)| > z$ for all r in $[\rho, 1]$ then $D_1 = \{x : |\nabla u_0| \leq t\} = D_0$ with $t = z$. On the other hand, if $|y_2'(1)| < z$ then we have $t < z$ to satisfy the condition $|D_1| = A$, in view of the continuity of the function $|y_2'(r)|$. In other words, D_1 should include a radially symmetric subset of D_0^c . This discussion proves our claim.

It remains to verify inequality (3.7). This is a standard result of the perturbation theory of eigenvalues that u_0 tends to ψ with $\|\psi\|_{L^2(\Omega)} = 1$ and λ converges to $\alpha\mu$ when β decreases to α [20]. The convergence of the eigenfunctions holds in the space $H_0^1(\Omega)$. Hence it yields that $y(r)$ and $y'(r)$ converge to $\psi(r)$ and $\psi'(r)$ almost everywhere in Ω , respectively. Since $y'(r)$ and $\psi'(r)$ are continuous functions on the sets $[0, \rho]$ and $[\rho, 1]$, the convergence is pointwise [16]. In summary, $|y_1'(r)|$ converges to $|\psi'(r)|$ pointwise for all r in $[0, \rho]$ and $|y_2'(r)|$ converges to $|\psi'(r)|$ pointwise in $[\rho, 1]$. Additionally, $||y_2'(\rho)| - |y_1'(\rho)||$ converges to zero when β approaches α . Invoking lemma 3.1, we see that $|\psi'(\rho)| - |\psi'(1)| = d_n > 0$ when $\rho > \rho_n$. Thus, if β is close to α enough, we have

$$||y_2'(\rho)| - |y_2'(1)|| > d_n/2, \quad (3.8)$$

and also

$$|y_2'(\rho)| \rightarrow |\psi'(\rho)|, \quad |y_2'(1)| \rightarrow |\psi'(1)|, \quad |y_2'(\rho)| \rightarrow |y_1'(\rho)|, \quad (3.9)$$

as β converges to α . Applying (3.8) and (3.9), leads us to inequalities

$$|y_2'(1)| < |y_1'(\rho)| \leq z.$$

□

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